Para-tt*-bundles on the tangent bundle of an almost para-complex manifold

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Abstract

In this paper we study para-tt^*-bundles (TM, D, S) on the tangent bundle of an almost para-complex manifold (M, τ). We characterise those para-tt^*-bundles with $\nabla = D + S$ induced by the one-parameter family of connections given by $\nabla^\theta = \exp(\theta\tau) \circ \nabla \circ \exp(-\theta\tau)$ and prove a uniqueness result for solutions with a para-complex connection D. Flat nearly para-Kähler manifolds and special para-complex manifolds are shown to be such solutions. We analyse which of these solutions admit metric or symplectic para-tt^*-bundles. Moreover, we give a generalisation of the notion of a para-pluriharmonic map to maps from almost para-complex manifolds (M, τ) into pseudo-Riemannian manifolds and associate to the above metric and symplectic para-tt^*-bundles generalised para-pluriharmonic maps into $\mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$, respectively into $SO_0(n, n)/U^\pi(C^n)$, where $U^\pi(C^n)$ is the para-complex analogue of the unitary group.

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Contents

1 Introduction

The subject of this work are para-tt^*-bundles (TM, D, S) on the tangent bundle an almost para-complex manifold (M, τ) as base. We generalise the results of the complex geometric setting which can be found in [S3]. In previous work we always supposed a para-complex base manifold. Motivated by the study of the complex case we admit non-integrable para-complex structures, since Levi-Civita flat *nearly para-Kähler manifolds* arise in this way as solutions of para-tt^*-bundles. Nearly para-Kähler manifolds as analogue of nearly Kähler manifolds were introduced in a recent paper of Ivanov and Zamkovoy [IZ]. A further class of solutions is given by special para-Kähler manifolds which were studied in [S2]. These geometries arise as one of the special geometries of Euclidean supersymmetry in [CMMS]. This means that para-tt^*-bundles can be seen as a common generalisation of these two geometries and give a kind of duality between them.
Now we discuss the structure of the paper. First we introduce the notion of (metric, symplectic) para-tt^*-bundles and describe these in terms of explicit geometric data. Part of this data is a family of flat connections D^θ on a vector bundle E with $\theta \in \mathbb{R}$ which is defined with help of a second connection D and a field $S \in \Gamma(T^*M \otimes \mathrm{End}(E))$. Given an almost para-complex manifold endowed with a flat connection ∇ we can consider a natural family of flat connections defined by

$$\nabla^\theta = \exp(\theta\tau) \circ \nabla \circ \exp(-\theta\tau) \text{ for } \theta \in \mathbb{R}.$$

We analyse para-tt^*-bundles, such that $D^\theta = \nabla^{\alpha\theta}$ for some $\alpha \in \mathbb{R}$. This choice is motivated by the study of solutions coming from special para-Kähler geometry.

If we restrict to para-tt^*-bundles carrying connections D which are para-complex, i.e. which satisfy $D\tau = 0$, then these para-tt^*-bundles can be recovered uniquely from the pair (∇, τ). Moreover, we give compatibility conditions on the pair (∇, τ). These conditions hold for *special para-complex* and nearly para-Kähler manifolds. We discuss two classes of solutions: The first corresponds to special para-complex manifolds with torsion and the second to almost para-complex manifolds endowed with a flat connection ∇ such that (∇, τ) satisfies the nearly para-Kähler condition (with torsion). Further we study whether these para-tt^*-bundles provide *metric* or *symplectic* para-tt^*-bundles. Examples of solutions of the first type are given by special para-Kähler manifolds and solutions of the second type arise on Levi-Civita flat nearly para-Kähler manifolds. In addition we show that neither the special para-Kähler condition is compatible with symplectic para-tt^*-bundles nor the nearly para-Kähler condition is compatible with solutions of metric para-tt^*-bundles.
There exists a relation between para-pluriharmonic maps and para-tt^*-geometry which was studied in [S2]. In this work we generalise the notion of a para-pluriharmonic map to maps from almost para-complex manifolds into pseudo-Riemannian manifolds. Then we introduce $\tilde{\mathbb{S}}^1$-pluriharmonic maps which generalise $\mathbb{S}^1$-pluriharmonic maps (cf. [S3]) to para-complex geometry. This is a generalisation of the notion of associated families (see for example [ET]). We relate these $\tilde{\mathbb{S}}^1$-pluriharmonic maps to the generalisation of para-pluriharmonic maps and prove a result, which relates generalised para-pluriharmonic maps to harmonic maps. With these preparations we are able to associate para-pluriharmonic maps into $\mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$, respectively into $SO_0(n,n)/U^\pi(C^n)$, to the above metric and symplectic para-tt^*-bundles. Here $U^\pi(C^n)$ is the para-complex analogue of the unitary group.
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2 Para-complex differential geometry

In this section we recall facts of para-complex differential geometry, which are needed in the later work. For more information we refer to [CMMS, IZ].

Definition 1 *A* para-complex structure *on a real finite dimensional vector space V is a non-trivial involution $\tau \in End(V)$ such that the two eigenspaces $V^\pm := ker(Id \mp \tau)$ to the eigenvalues ± 1 of τ have the same dimension. We call the pair (V, τ) a* para-complex vector-space.
An almost para-complex structure *on a smooth manifold M is an endomorphism field $\tau \in \Gamma(End(TM))$ such that, for all $p \in M$, τ_p is a para-complex structure on T_pM. It is called* integrable *if the distributions $T^\pm M = ker(Id \mp \tau)$ are integrable. An integrable almost para-complex structure on M is called a* para-complex structure on M *and a manifold M endowed with a para-complex structure is called a* para-complex manifold.

The integrability of a para-complex structure is obstructed (cf. [CMMS]) by the vanishing of a tensor, also called Nijenhuis tensor, which is defined as

$$N_\tau(X, Y) := [X, Y] + [\tau X, \tau Y] - \tau[X, \tau Y] - \tau[\tau X, Y].$$

The decomposition of the tangent bundle of an (almost) para-complex manifold M into the eigenspaces $T^{\pm}M$ extends to a bi-grading on the exterior algebra:

$$\Lambda^k T^*M = \bigoplus_{k=p+q} \Lambda^{p+,q-} T^*M \tag{2.1}$$

and induces an obvious bi-grading on exterior forms with values in a vector bundle E.

The para-complex numbers are the real algebra, which is generated by 1 and by the para-complex unit e with $e^2 = 1$. We denote the para-complex numbers by the symbol C.
For all $z = x + ey \in C$ with $x, y \in \mathbb{R}$ we define the para-complex conjugation $\bar{\cdot} : C \to C, x + ey \mapsto x - ey$ and the real and imaginary parts of z by

$$\Re(z) := \frac{z + \bar{z}}{2} = x,\ \Im(z) := \frac{e(z - \bar{z})}{2} = y.$$

The free C-module C^n is a para-complex vector space where its para-complex structure is the multiplication with e and the para-complex conjugation of C extends to $\bar{\cdot} : C^n \to C^n$, $v \mapsto \bar{v}$.
Note the identity $z\bar{z} = x^2 - y^2$. Therefore the algebra C is sometimes called the hyper-complex numbers. The circle $\mathbb{S}^1 = \{z = x + iy \in \mathbb{C} \,|\, x^2 + y^2 = 1\}$ is replaced by the four hyperbolas $\{z = x + ey \in C \,|\, x^2 - y^2 = \pm 1\}$. $\tilde{\mathbb{S}}^1$ is defined to be the hyperbola given by the one parameter group $\{z(\theta) = \cosh(\theta) + e\sinh(\theta) \,|\, \theta \in \mathbb{R}\}$.
The para-complex dimension of a para-complex manifold M is the integer $n = \dim_C M := \frac{1}{2}\dim_{\mathbb{R}} M$.
Now we consider the para-complexification $TM^C = TM \otimes_{\mathbb{R}} C$ of the tangent bundle TM of an almost para-complex manifold (M, τ) and extend $\tau : TM \to TM$ C-linearly to $\tau : TM^C \to TM^C$. Then for all $p \in M$ the free C-module T_pM^C decomposes as C-module into the direct sum of two free C-modules

$$T_pM^C = T_p^{1,0}M \oplus T_p^{0,1}M, \tag{2.2}$$

where $T_p^{1,0}M := \{X + e\tau X | X \in T_pM\}$ and $T_p^{0,1}M := \{X - e\tau X | X \in T_pM\}$. The subbundles $T_p^{1,0}M$ and $T_p^{0,1}M$ can be characterised as the $\pm e$-eigenbundles of the map $\tau : TM^C \to TM^C$, i.e. $\tau = e$ on $T^{1,0}M$ and $\tau = -e$ on $T^{0,1}M$.
In the same manner we decompose $T^*M^C = \Lambda^{1,0}T^*M \oplus \Lambda^{0,1}T^*M$ into the $\pm e$-eigenbundles of the dual para-complex structure $\tau^* : T^*M^C \to T^*M^C$. This decomposition induces a bi-grading on the C-valued exterior forms

$$\Lambda^k T^*M^C = \bigoplus_{k=p+q} \Lambda^{p,q} T^*M$$

and finally on the C-valued differential forms on M

$$\Omega^k_C(M) = \bigoplus_{k=p+q} \Omega^{p,q}(M).$$

We remark, that for the case $(1,1)$ and $(1+, 1-)$ the two gradings induced by τ coincide in following sense: $\Lambda^{1,1}\, T^*M = (\Lambda^{1+,1-}\, T^*M) \otimes C$.
We need to introduce some additional notions.

Definition 2 *Let (V, τ) be a para-complex vector space. A* para-hermitian *scalar product g on V is a pseudo-Euclidean scalar product for which τ is an anti-isometry, i.e.*

$$\tau^* g = g(\tau \cdot, \tau \cdot) = -g(\cdot, \cdot).$$

A para-hermitian vector space *(V, τ, g) is a para-complex vector space (V, τ) endowed with a para-hermitian scalar product g. The pair (τ, g) is called* para-hermitian structure *on the vector space V.*

Definition 3 *Let (V, τ, g) be a para-hermitian vector space. The para-unitary group of V is the automorphism group*

$$U^\pi(V) := Aut(V, \tau, g) = \{L \in GL(V) \,|\, [L, \tau] = 0 \text{ and } L^* g = g\}.$$

Definition 4 *An* almost para-hermitian manifold *(M, τ, g) is an almost para-complex manifold (M, τ) endowed with a pseudo-Riemannian metric g such that $\tau^* g = -g$. If τ is integrable, we call (M, τ, g) a* para-hermitian manifold*. The two-form $\omega := g(\tau \cdot, \cdot)$ is called the* fundamental two-form *of the almost para-hermitian manifold (M, τ, g).*

3 Para-tt^*-bundles

We generalise the notion of a para-tt^*-bundle which was introduced in [S2], by admitting a base manifold (M, τ) with a non-integrable para-complex structure τ. In this way nearly para-Kähler manifolds appear as solutions of para-tt^*-geometry on the tangent bundle TM. Further we introduce symplectic para–tt^*-bundles.

Definition 5 *A* para-tt^*-bundle *or* ptt^*-bundle *(E, D, S) over an almost para-complex manifold (M, τ) is a real vector bundle $E \to M$ endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \mathrm{End}\, E)$ which satisfy the* ptt^*-equation

$$R^\theta = 0 \quad \text{for all} \quad \theta \in \mathbb{R}, \tag{3.1}$$

where R^θ is the curvature tensor of the connection D^θ defined by

$$D^\theta_X := D_X + \cosh(\theta) S_X + \sinh(\theta) S_{\tau X} \quad \text{for all} \quad X \in TM. \tag{3.2}$$

A metric ptt^*-bundle *(E, D, S, g) is a ptt^*-bundle (E, D, S) endowed with a possibly indefinite D-parallel fiber metric g such that for all $p \in M$*

$$g(S_X Y, Z) = g(Y, S_X Z) \quad \text{for all} \qquad X, Y, Z \in T_pM. \tag{3.3}$$

A symplectic ptt^*-bundle *(E, D, S, ω) is a ptt^*-bundle (E, D, S) endowed with the structure of a symplectic vector bundle*[1] *(E, ω), such that ω is D-parallel and S is ω-symmetric, i.e. for all $p \in M$*

$$\omega(S_X Y, Z) = \omega(Y, S_X Z) \quad \text{for all} \qquad X, Y, Z \in T_pM. \tag{3.4}$$

[1] see D. Mc Duff and D. Salamon [McDS]

Remark 1
If (E, D, S) is a ptt-bundle then (E, D, S^θ) is a ptt*-bundle for all $\theta \in \mathbb{R}$, where*

$$S^\theta := D^\theta - D = \cosh(\theta) S + \sinh(\theta) S_\tau \,. \tag{3.5}$$

The same remark applies to metric ptt-bundles and symplectic ptt*-bundles.*

Like for *ptt**-bundles (E, D, S) over a para-complex manifold we find explicit equations for D and S.

Proposition 1 *Let E be a real vector bundle over an almost para-complex manifold (M, τ) endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \operatorname{End} E)$.*
Then (E, D, S) is a ptt-bundle if and only if D and S satisfy the following equations:*

$$R^D + S \wedge S = 0, \tag{3.6}$$
$$S \wedge S \text{ is of type (1,1)}, \tag{3.7}$$
$$[D_X, S_Y] - [D_Y, S_X] - S_{[X,Y]} = 0, \quad \forall\, X, Y \in \Gamma(TM), \tag{3.8}$$
$$[D_X, S_{\tau Y}] - [D_Y, S_{\tau X}] - S_{\tau[X,Y]} = 0, \quad \forall\, X, Y \in \Gamma(TM). \tag{3.9}$$

Fixing a torsion-free connection on (M, τ) the last two equations are equivalent to

$$d^D S = 0 \quad \text{and} \quad d^D S_\tau = 0. \tag{3.10}$$

Proof: (compare [S2]) As for *ptt**-bundles over para-complex manifolds (M, τ) one calculates using the theorems of addition $2\cosh(\theta)\sinh(\theta) = \sinh(2\theta)$, $2\cosh^2(\theta) = 1 + \cosh(2\theta)$ and $2\sinh^2(\theta) = \cosh(2\theta) - 1$, the (finite) decomposition of R^θ in $1, \cosh(n\theta)$ and $\sinh(n\theta)$, for $n = 1, 2$. The *ptt**-equation $R^\theta = 0$ is equivalent to the vanishing of all components. This yields the claimed equations. □

4 Solutions on the tangent bundle of an almost para-complex manifold

In this section we consider para-complex manifolds (M, τ) endowed with a flat connection ∇. It is natural to regard the one-parameter family of connections ∇^θ, which is defined by

$$\nabla^\theta_X Y = \exp(\theta\tau) \nabla_X (\exp(-\theta\tau) Y) \text{ for } X, Y \in \Gamma(TM), \tag{4.1}$$

where $\exp(\theta\tau) = \cosh(\theta) Id + \sinh(\theta)\tau$. The connection ∇ is flat if and only if ∇^θ is flat.

Definition 6 *Two one-parameter families of connections ∇^θ and D^θ, with $\theta \in \mathbb{R}$, on some vector bundle E are called* (linear) equivalent with factor $\alpha \in \mathbb{R}$ *if they satisfy the equation $\nabla^\theta = D^{\alpha\theta}$.*

In the sequel we study *ptt**-bundles (TM, D, S) such that the connection D^θ defined in equation (3.2) is linear equivalent to the connection ∇^θ defined in equation (4.1). This ansatz is motivated by our previous study of *ptt**-bundles coming from special para-complex and special para-Kähler manifolds (see [S2]).

Proposition 2 *Given an almost para-complex manifold (M,τ) with a flat connection ∇. Then a decomposition of $\nabla = D + S$ in a connection D and a section S in $T^*M \otimes$ End (TM) defines a ptt*-bundle (TM, D, S), such that the family of connections D^θ is linear equivalent to the family of connections ∇^θ with factor $\alpha = \pm 2$ if and only if S and D satisfy*

$$S_{\tau X} = \pm \tau S_X Y$$

and

$$-(D_X\tau)Y = \tau S_X Y + S_X \tau Y =: \{S_X, \tau\}Y$$

for all $X, Y \in \Gamma(TM)$.

Proof: One has to compute ∇^θ for $X, Y \in \Gamma(TM)$:

$$\begin{aligned}
\nabla^\theta_X Y &= \exp(\theta\tau)(D_X + S_X)(\cosh(\theta)Id - \sinh(\theta)\tau)Y) \\
&= D_X Y - \exp(\theta\tau)\sinh(\theta)(D_X\tau)Y \\
&+ (\cosh(\theta)Id + \sinh(\theta)\tau)S_X(\cosh(\theta)Id - \sinh(\theta)\tau)Y \\
&= D_X Y - (\cosh(\theta)\sinh(\theta) + \sinh^2(\theta)\tau)(D_X\tau)Y + \cosh^2(\theta)S_X Y \\
&- \sinh^2(\theta)\tau S_X \tau Y - \cosh(\theta)\sinh(\theta)[S_X, \tau]Y.
\end{aligned}$$

This yields with the theorems of addition, i.e.

$$2\sinh(\theta)\cosh(\theta) = \sinh(2\theta), \quad 2\cosh^2(\theta) = 1+\cosh(2\theta) \quad \text{and} \quad 2\sinh^2(\theta) = \cosh(2\theta)-1,$$

the identity

$$\begin{aligned}
\nabla^\theta_X Y &= D_X Y - \frac{1}{2}\sinh(2\theta)(D_X\tau)Y - \frac{1}{2}(\cosh(2\theta) - 1)\tau(D_X\tau)Y \\
&+ \frac{1}{2}(1 + \cosh(2\theta))S_X Y - \frac{1}{2}(\cosh(2\theta) - 1)\tau S_X \tau Y - \frac{1}{2}\sinh(2\theta)[S_X, \tau]Y \\
&= D_X Y + \frac{1}{2}\left[S_X + \tau S_X \tau + \tau D_X \tau\right] Y \\
&+ \frac{1}{2}\sinh(2\theta)\left[[\tau, S_X] - D_X\tau\right] Y \\
&+ \frac{1}{2}\cosh(2\theta)\left[S_X - \tau S_X \tau - \tau D_X \tau\right] Y \\
&\overset{!}{=} D_X Y + \cosh(\vartheta)T_X Y + \sinh(\vartheta)T_{\tau X} Y \text{ with } \vartheta = \pm 2\theta,
\end{aligned}$$

where we have to determine $T \in \Gamma(T^*M \otimes \text{End}\,(TM))$.
Comparing coefficients of $1, \cosh(2\theta), \sinh(2\theta)$ gives

$$\begin{aligned}
-\tau(D_X\tau)Y &= S_X Y + \tau S_X \tau Y, \text{ or equivalentely} && (4.2)\\
-(D_X\tau)Y &= \tau S_X Y + S_X \tau Y = \{S_X, \tau\}Y, \\
T_X Y &= \frac{1}{2}(S_X Y - \tau S_X \tau Y - \tau(D_X\tau)Y) \overset{(4.2)}{=} S_X Y, && (4.3)\\
T_{\tau X} Y &= \pm\frac{1}{2}([\tau, S_X]Y - (D_X\tau)Y) \\
&\overset{(4.2)}{=} \pm\frac{1}{2}(\tau S_X Y - S_X \tau Y + \tau S_X Y + S_X \tau Y) = \pm\tau S_X Y. && (4.4)
\end{aligned}$$

The constraint on S follows from the last two equations, i.e.

$$S_{\tau X} = \pm\tau S_X Y$$

and the one on D and S from the first equation. $\square$

We further specialise to para-complex connections D, i.e. connections D satisfying the identity $D\tau = 0$. The existence of such connections was provided by Theorem 1 of [S2].

Corollary 1 *Let (M, τ) be an almost para-complex manifold endowed with a flat connection ∇ and let $\nabla = D + S$ be a given decomposition in a connection D and a section S in $T^*M \otimes \mathrm{End}\,(TM)$, such that τ is D-parallel. Then (TM, D, S) defines a ptt*-bundle, such that the family of connections D^θ is linear equivalent to the family of connections ∇^θ with factor $\alpha = \pm 2$ if and only if S satisfies*

$$S_{\tau X} = \pm\tau S_X \textit{ and } \{S_X, \tau\} = 0.$$

Proof: The second condition in proposition 2 yields using $D\tau = 0$ the equation $\{S_X, \tau\} = 0$. The first constraint of proposition 2 remains $S_{\tau X} = \pm\tau S_X \overset{\{S_X,\tau\}=0}{=} \mp S_X \tau$. $\square$

To establish a uniqueness result we need the next lemma.

Lemma 1 *Let (M, τ) be an almost para-complex manifold. Let a connection ∇ on M be given. Then the connection D and the section S of $T^*M \otimes \mathrm{End}\,(TM)$ defined by*

$$S_X Y = -\frac{1}{2}\tau(\nabla_X \tau)Y \textit{ and } D_X Y = \nabla_X Y - S_X Y \textit{ for } X, Y \in \Gamma(TM), \tag{4.5}$$

satisfy $D\tau = 0$ and $\{S_X, \tau\} = 0$.
Otherwise, suppose that ∇ decomposes as $\nabla = D + S$, where D is a connection on M and S is a section in $T^*M \otimes \mathrm{End}\,(TM)$, such that τ is D-parallel, i.e. $D\tau = 0$ and S anti-commutes with τ, i.e. $\{S_X, \tau\} = 0$ for all $X \in \Gamma(TM)$. Then D and S are uniquely given by equation (4.5).

Proof: We check $\nabla = D + S$ and $S_X \tau Y = -\frac{1}{2}\tau(\nabla_X \tau)\tau Y = \frac{1}{2}\tau^2(\nabla_X \tau)Y = -\tau S_X Y$, where the second equality follows from deriving $\tau^2 = Id$. In addition it is

$$(D_X \tau)Y = (\nabla_X \tau)Y - [S_X, \tau] \overset{\{S_X,\tau\}=0}{=} (\nabla_X \tau)Y + 2\tau S_X = 0.$$

It rests to show the uniqueness: One starts with D' and S' having same properties and computes

$$0 = (D'_X \tau)Y = (\nabla_X \tau)Y - [S'_X, \tau]Y = (\nabla_X \tau)Y + 2\tau S'_X Y.$$

This further implies

$$S'_X Y = -\frac{1}{2}\tau(\nabla_X \tau)Y = S_X Y \text{ and } D'_X Y = \nabla_X Y - S'_X Y = \nabla_X Y - S_X Y = D_X Y.$$

$\square$

Corollary 1 and lemma 1 imply the following uniqueness result:

Theorem 1 *Let (M, τ) be an almost para-complex manifold endowed with a flat connection ∇. If a decomposition of $\nabla = D + S$ in a connection D and a section S in $T^*M \otimes \mathrm{End}\,(TM)$, such that τ is D-parallel, i.e. $D\tau = 0$, defines a ptt*-bundle (TM, D, S), such that the family of connections D^θ is linear equivalent to the the family of connections ∇^θ with factor $\alpha = \pm 2$, then D and S are uniquely determined by the equations $S = -\frac{1}{2}\tau(\nabla\tau)$ and $D = \nabla - S$.*
Moreover, (TM, D, S) as above defines a ptt-bundle, such that the family of connections D^θ is linear equivalent to the family of connections ∇^θ with factor $\alpha = \pm 2$, if and only if τ satisfies $(\nabla_{\tau X}\tau) = \pm\tau(\nabla_X\tau)$ and D and S are given by equation (4.5).*

Some classes of examples which satisfy the condition $S_{\tau X} = \pm\tau S_X$ are given in the following propositions.

Proposition 3 *Let an almost para-complex manifold (M, τ) endowed with a connection ∇ be given and let S be the section in $T^*M \otimes \mathrm{End}\,(TM)$ defined by*

$$S := -\frac{1}{2}\tau(\nabla\tau). \tag{4.6}$$

If the pair (∇, τ) satisfies one of the following conditions

(i) (∇, τ) *is* special, *i.e.* $(\nabla_X\tau)Y = (\nabla_Y\tau)X$ *for all* $X, Y \in \Gamma(TM)$,

(ii) (∇, τ) *satisfies the* nearly para-Kähler condition, *i.e.* $(\nabla_X\tau)Y = -(\nabla_Y\tau)X$ *for all* $X, Y \in \Gamma(TM)$,

then it holds $S_{\tau X}Y = -\tau S_X Y$.

Proof: We prove (i) and (ii) in the same calculation:

$$(\nabla_{\tau X}\tau)Y \;=\; \pm(\nabla_Y\tau)\tau X = \mp\tau(\nabla_Y\tau)X = -\tau(\nabla_X\tau)Y.$$

This implies $S_{\tau X}Y = -\frac{1}{2}\tau(\nabla_{\tau X}\tau)Y = \frac{1}{2}\tau^2(\nabla_X\tau)Y = -\tau S_X Y$.
□

Proposition 4 *Let a para-complex manifold (M, τ) endowed with a connection ∇ be given and let S be the section in $T^*M \otimes \mathrm{End}\,(TM)$ defined by*

$$S := -\frac{1}{2}\tau(\nabla\tau). \tag{4.7}$$

If ∇ is (anti-)adapted, *i.e.* $\nabla_{\tau X} = \pm\tau\nabla_X Y$ *for all para-holomorphic vector-fields X, Y, then it holds $S_{\tau X}Y = \pm\tau S_X Y$.*

Proof: Since the connection ∇ is (anti-)adapted, we obtain for all para-holomorphic vector-fields X, Y :

$$(\nabla_{\tau X}\tau)Y = \pm\tau(\nabla_X\tau)Y.$$

The following computation finishes the proof

$$S_{\tau X}Y = -\frac{1}{2}\tau(\nabla_{\tau X}\tau)Y = \mp\frac{1}{2}\tau^2(\nabla_X\tau)Y = \pm\tau S_X Y.$$

$\square$

Remark 2 *One observes that in proposition 3 the condition (i) is the symmetry of $S_X Y$ and condition (ii) is its anti-symmetry. We recall that if the connection ∇ is torsion-free, flat and special then (M, τ, ∇) is a* special para-complex manifold*, see [CMMS, S2]. Ptt*-bundles coming from special para-complex manifolds and special para-Kähler manifolds were studied in [S2].*
Moreover we remark, that the second condition arises in nearly para-Kählerian geometry (compare [IZ]) and therefore is quite natural. These geometries as solutions of ptt*-geometry are discussed later in this work.
Finally, the notion of adapted connections appeared in the study of decompositions on (para-holomorphic) para-complex vector bundles, compare [LS].

5 Solutions on almost para-hermitian manifolds

In this section we study the question under which assumptions almost para-complex manifolds (M, τ) endowed with a flat connection ∇ such that (∇, τ) is special or satisfies the nearly para-Kähler condition define symplectic or metric *ptt**-bundles.

First we recall a lemma from tensor-algebra:

Lemma 2 *Let V be a vector-space and $\alpha \in T^3(V^*)$ an element in the third tensorial power of V^*, the dual space of V. Suppose that $\alpha(X, Y, Z)$ is symmetric (resp. anti-symmetric) in X, Y and Y, Z and $\alpha(X, Y, Z)$ is anti-symmetric (resp. symmetric) in X, Z then $\alpha = 0$.*

Proof: It is $\alpha(X, Y, Z) = \epsilon\alpha(Y, X, Z) = \epsilon\alpha(X, Z, Y)$ with $\epsilon \in \{\pm 1\}$ which implies $\alpha(X, Y, Z) = \epsilon\alpha(Y, X, Z) = \epsilon^2\alpha(Y, Z, X) = \epsilon^3\alpha(Z, Y, X)$. But further it holds $\alpha(X, Y, Z) = -\epsilon\alpha(Z, Y, X)$ and consequently $-\alpha(Z, Y, X) = \epsilon^2\alpha(Z, Y, X) = \alpha(Z, Y, X)$. This shows $\alpha = 0$. $\square$

Proposition 5 *Let (M, τ) be an almost para-complex manifold endowed with a flat connection ∇, such that (∇, τ) is special. Define S, a section in $T^*M \otimes \mathrm{End}\,(TM)$, by*

$$S := -\frac{1}{2}\tau(\nabla\tau), \tag{5.1}$$

then $(TM, D = \nabla - S, S)$ defines a ptt*-bundle. If in addition (TM, D, S, ω) is a symplectic ptt*-bundle, then it is trivial, i.e. $S = 0$.

Proof: Due to theorem 1 and proposition 3 (TM, D, S) is a *ptt**-bundle.
Suppose, that (TM, D, S, ω) is a symplectic *ptt**-bundle. In order to prove the second part of the proposition, we define the tensor

$$\alpha(X, Y, Z) := \omega(S_X Y, Z) = g(\tau S_X Y, Z).$$

$\alpha(X, Y, Z)$ is symmetric in X, Y, since (∇, τ) is special, i.e. $\nabla\tau$ is symmetric in X, Y. Moreover, it holds

$$\begin{aligned} \alpha(X, Y, Z) &= \omega(S_X Y, Z) = -\omega(Z, S_X Y) \\ &= -\omega(Z, S_Y X) = -\omega(S_Y Z, X) = -\omega(S_Z Y, X) = -\alpha(Z, Y, X). \end{aligned}$$

This is the anti-symmetry of $\alpha(X, Y, Z)$ in X, Z. Finally we compute

$$\begin{aligned} \alpha(X, Y, Z) &= \omega(S_X Y, Z) = \omega(Y, S_X Z) \\ &= \omega(Y, S_Z X) = -\omega(S_Z X, Y) = -\alpha(Z, X, Y) = -\alpha(X, Z, Y), \end{aligned}$$

i.e. the anti-symmetry of $\alpha(X, Y, Z)$ in Y, Z.
Hence α vanishes and consequently S is zero. □

Proposition 6 *Let an almost para-complex manifold (M, τ) endowed with a flat connection ∇, such that (∇, τ) satisfies the nearly para-Kähler condition be given. Define S, a section S in $T^*M \otimes \mathrm{End}\,(TM)$, by*

$$S := -\frac{1}{2}\tau(\nabla\tau), \tag{5.2}$$

then $(TM, D = \nabla - S, S)$ defines a ptt*-bundle. Suppose, that (TM, D, S, g) is a metric ptt*-bundle, then it is trivial, i.e. $S = 0$.

Proof: Due to theorem 1 and proposition 3 (TM, D, S) is a *ptt**-bundle.
Suppose, that it is a metric *ptt**-bundle. The proof is obtained by analysing the symmetries of the tensor

$$\alpha(X, Y, Z) := g(S_X Y, Z).$$

$\alpha(X, Y, Z)$ is anti-symmetric in X, Y, since by the nearly para-Kähler condition $\nabla\tau$ is anti-symmetric in X, Y. In addition it holds

$$\begin{aligned} \alpha(X, Y, Z) &= g(S_X Y, Z) = g(Z, S_X Y) \\ &= -g(Z, S_Y X) = -g(S_Y Z, X) = g(S_Z Y, X) = \alpha(Z, Y, X), \end{aligned}$$

which is the symmetry of $\alpha(X, Y, Z)$ in X, Z. Finally one has

$$\begin{aligned} \alpha(X, Y, Z) &= g(S_X Y, Z) = g(Y, S_X Z) \\ &= -g(Y, S_Z X) = -g(S_Z X, Y) = -\alpha(Z, X, Y) = \alpha(X, Z, Y), \end{aligned}$$

i.e. the symmetry of $\alpha(X, Y, Z)$ in Y, Z.
Hence α vanishes by the above lemma and so does S. □

The following theorem gives solutions of symplectic para-*tt**-bundles on the tangent bundle, which are more general then the later discussed nearly para-Kähler manifolds in the sense, that we admit the connection ∇ to have torsion, but more special in the sense, that our connection ∇ has to be flat.

Theorem 2 *Let (M, τ, g) be an almost para-hermitian manifold endowed with a flat metric connection ∇, such that the pair (∇, τ) satisfies the nearly para-Kähler condition. Define S, a section in $T^*M \otimes \mathrm{End}\,(TM)$, by*

$$S := -\frac{1}{2}\tau(\nabla\tau), \tag{5.3}$$

then $(TM, D = \nabla - S, S, \omega = g(\tau\cdot, \cdot))$ defines a symplectic ptt-bundle.*
The torsion T^∇ of the connection ∇ and T^D of the connection D are related by

$$T^\nabla - T^D = 2S$$

and it holds $D\tau = 0$.

Proof: Due to theorem 1 and proposition 3 (TM, D, S) is a *ptt**-bundle.
It remains to check $D\omega = 0$ and that S is ω-symmetric.
We observe, that, as g is para-hermitian and $\nabla g = 0$, $\nabla_X\tau$ is skew-symmetric with respect to g. This yields by the following calculation, that S is skew-symmetric with respect to g:

$$\begin{aligned} -2g(S_XY, Z) &= g(\tau(\nabla_X\tau)Y, Z) = -g((\nabla_X\tau)Y, \tau Z) \\ &= g(Y, (\nabla_X\tau)\tau Z) = -g(Z, \tau(\nabla_X\tau)Y) = 2g(Y, S_XZ). \end{aligned}$$

The definition of $\omega = g(\tau\cdot, \cdot)$ and $\{S_X, \tau\} = 0$ yield the ω-symmetry of S_X.
Further it holds $D = \nabla + \frac{1}{2}\tau\nabla\tau$, which implies

$$D\tau = \nabla\tau + \frac{1}{2}[\tau\nabla\tau, \tau] = 0.$$

This shows $D\omega = 0$ if and only if $Dg = 0$. But $\nabla g = 0$ and S is skew-symmetric with respect to g, so g is parallel for $D = \nabla - S$.
This shows that $(TM, D = \nabla - S, S, \omega)$ is a symplectic *ptt**-bundle.
From $\nabla = D + S$ we obtain for the torsions

$$T^\nabla(X, Y) = T^D(X, Y) + S_XY - S_YX = T^D(X, Y) + 2S_XY.$$

□

We recall the definition of special para-complex and special para-Kähler manifolds (see [CMMS, S2]):

Definition 7 *A special para-Kähler manifold consists of the data (M, τ, g, ∇) where (M, τ, g) is a para-Kähler manifold with ∇-parallel para-Kähler form and (M, τ, ∇) is* a special para-complex manifold, *i.e. (M, τ) is a para-complex manifold endowed with a flat and torsion-free connection ∇ such that (∇, τ) is special.*

The next theorem gives solutions of metric para-*tt**-bundles on the tangent bundle, which are more general then special para-Kähler manifolds in the sense, that we admit connections ∇ with torsion.

Theorem 3 *Let an almost para-hermitian manifold (M, τ, g) endowed with a flat connection ∇ be given. Suppose that (∇, τ) is special and the two-form $\omega = g(\tau\cdot, \cdot)$ is ∇-parallel. Define S, a section in $T^*M \otimes \mathrm{End}\,(TM)$, by*

$$S := -\frac{1}{2}\tau(\nabla\tau), \tag{5.4}$$

then $(TM, D = \nabla - S, S, g)$ defines a metric ptt-bundle. Moreover, the connections D and ∇ have the same torsion and we have $D\tau = 0$.*
If in addition ∇ is torsion-free, then D is the Levi-Civita connection of g, (M, τ, g) is a para-Kähler manifold and (M, τ, g, ∇) is a special para-Kähler manifold.

Proof: Due to theorem 1 and proposition 3 (TM, D, S) defines a *ptt**-bundle.
It remains to prove $Dg = 0$ and that S is g-symmetric.
First we remark that $\omega(\tau X, Y) = -\omega(X, \tau Y)$ as g is para-hermitian. This implies by $\nabla\omega = 0$ the ω-skew-symmetry of $\nabla_X\tau$, which yields that $S_X = -\frac{1}{2}\tau(\nabla_X\tau)$ is ω-skew-symmetric, since $\tau(\nabla_X\tau) = -(\nabla_X\tau)\tau$. Finally $\{S_X, \tau\} = 0$ shows the g-symmetry of S_X.
Moreover, we compute

$$D\tau = \nabla\tau + \frac{1}{2}[\tau\nabla\tau, \tau] = 0$$

and consequently $Dg = 0$ is equivalent to $D\omega = 0$.
From $\nabla\omega = 0$ and the ω-skew-symmetry of S it follows $D\omega = (\nabla - S)\omega = 0$.
From the symmetry of $\nabla\tau$, i.e. $(\nabla_X\tau)Y = (\nabla_Y\tau)X$ for all $X, Y \in TM$ we obtain $S_XY = S_YX$. This shows, that the torsions of the connections ∇ and D coincide.
Suppose now that ∇ is torsion-free. This implies, that $D = \nabla - S$ is torsion-free and consequently the Levi-Civita-connection of g. Further the equation $\nabla\omega = 0$ implies $d\omega = 0$, since ∇ is torsion-free. As $D\tau = 0$ and D is torsion-free, τ is integrable. Hence (M, τ, g) is para-Kähler. In addition (M, τ, ∇) is special para-complex by the conditions on ∇ and τ. Summarising (M, τ, g, ∇) is a special para-Kähler manifold. □

In [S2] we treated special para-Kähler solutions of *ptt**-geometry in more details.

Now we are going to apply the above results to nearly para-Kähler manifolds. In order to do this we recall the notion of a nearly para-Kähler manifold which was recently introduced by Ivanov and Zamkovoy [IZ].

Definition 8 *An almost para-hermitian manifold (M, τ, g) is called* nearly para-Kähler manifold*, if its Levi-Civita connection $\nabla = \nabla^g$ satisfies the equation*

$$(\nabla_X\tau)Y = -(\nabla_Y\tau)X, \quad \forall X, Y \in \Gamma(TM). \tag{5.5}$$

A nearly para-Kähler manifold is called strict*, if $\nabla\tau \neq 0$.*

We recall that the tensor $\nabla\tau$ defines two tensors A and B

$$A(X, Y, Z) := g((\nabla_X\tau)Y, Z) \text{ and } B(X, Y, Z) := g((\nabla_X\tau)Y, \tau Z) \text{ with } X, Y, Z \in TM,$$

which are both (real) three-forms of type $(3,0)+(0,3)$.
A connection of particular importance in nearly para-Kähler geometry is the connection $\bar{\nabla}$ defined by

$$\bar{\nabla}_X Y := \nabla_X Y - \frac{1}{2}(\nabla_X \tau)\tau Y, \text{ for all } X, Y \in \Gamma(TM). \tag{5.6}$$

We remark, that $\bar{\nabla}$ is the unique connection with totally skew-symmetric torsion satisfying $\bar{\nabla} g = 0$ and $\bar{\nabla}\tau = 0$ (compare [IZ]).
The torsion of the connection $\bar{\nabla}$ is given by

$$T^{\bar{\nabla}}(X,Y) = -(\nabla_X \tau)\tau Y, \text{ for all } X, Y \in \Gamma(TM) \tag{5.7}$$

and it vanishes if and only if (M,τ,g) is a para-Kähler manifold.

Corollary 2 *Let (M,τ,g) be a nearly para-Kähler manifold, such that its Levi-Civita connection ∇ is flat and let S be the section in $T^*M \otimes \mathrm{End}\,(TM)$ defined by*

$$S := -\frac{1}{2}\tau(\nabla\tau), \tag{5.8}$$

then $(TM,\bar{\nabla},S)$ defines a ptt*-bundle. Suppose, that $(TM,\bar{\nabla},S,g)$ is a metric ptt*-bundle, then it is trivial, i.e. $S=0$ and consequently (M,τ,g) is para-Kähler.

Proof: If one puts $D=\bar{\nabla}$ we are in the situation of proposition 6. □

Remark 3 *In common work with V. Cortés [CS2] we gave a constructive classification of flat nearly pseudo-Kähler manifolds. The application of the methods of [CS2] to nearly para-Kähler manifolds is work in progress [CS3].*

Theorem 4 *Let (M,τ,g) be a nearly para-Kähler manifold, such that its Levi-Civita connection ∇ is flat. Let S be the section in $T^*M \otimes \mathrm{End}\,(TM)$ defined by*

$$S := -\frac{1}{2}\tau(\nabla\tau), \tag{5.9}$$

then $(TM,\bar{\nabla},S,\omega := g(\tau\cdot,\cdot))$ is a symplectic ptt*-bundle. Further it holds

$$B(X,Y,Z) = 2g(S_X Y, Z) \text{ and } \bar{\nabla}\tau = 0. \tag{5.10}$$

Proof: By setting $D=\bar{\nabla}$ we are in the situation of theorem 2. In addition it holds

$$2g(S_X Y, Z) \;=\; -g(\tau(\nabla_X\tau)Y, Z) = g((\nabla_X\tau)Y, \tau Z) = B(X,Y,Z).$$

□

6 Para-pluriharmonic maps from almost para-complex manifolds into pseudo-Riemannian manifolds

In this section we generalise the notion of a para-pluriharmonic map to maps from almost para-complex manifolds to pseudo-Riemannian manifolds and we introduce the para-complex analogue of an associated family (compare [ET] for the complex setting). Afterwards we show that maps admitting the para-complex analogue of an associated family give rise to a para-pluriharmonic map and we give conditions under which a para-pluriharmonic map is harmonic.

Let (M, τ) be an almost para-complex manifold of real dimension $2n$. In complex geometry it is well-known (compare [KN]) that on every almost complex manifold (M, J) there exists a connection with torsion $T = \frac{1}{4}N_J$ where N_J is the Nijenhuis tensor of the complex struture J. This result was generalised to para-complex geometry in Theorem 1 of [S2], which states that on an almost para-complex manifold (M, τ) there exists a para-complex connection with torsion $T = -\frac{1}{4}N_\tau$ where N_τ is the Nijenhuis tensor of the para-complex struture τ.

Definition 9 *Let (M, τ) be an almost para-complex manifold. A connection D on the tangent bundle of M is called* nice *if it is para-complex and its torsion T satisfies $T = \lambda N_\tau$ with a smooth function λ on M.*

We introduce the notion of a para-pluriharmonic map from an almost para-complex manifold:

Definition 10 *Let (M, τ, D) be an almost para-complex manifold endowed with a nice connection D on TM and N be a smooth manifold endowed with a connection ∇^N. Denote by ∇ the connection on $T^*M \otimes f^*TN$ which is induced by D and ∇^N.*
A smooth map $f : M \to N$ is called para-pluriharmonic *if and only if it satisfies the equation*

$$(\nabla df)^{1,1} = 0. \tag{6.1}$$

Remark 4 *First we remark, that for a para-complex manifold (M, τ) and a pseudo-Riemannian target manifold (N, h) with its Levi-Civita connection ∇^h the para-pluriharmonic equation* (6.1) *does not depend on the connection D if D is chosen in an appropriate class (compare [S2]). In fact nice connections on para-complex manifolds belong to this class. A famous case are para-Kähler manifolds (M, τ, g), where D is taken to be the Levi-Civita connection.*

To deal with associated families of para-pluriharmonic maps we need to recall an integrability condition satisfied by the differential of a smooth map. Denote by N a smooth manifold with a connection ∇^N on its tangent bundle having torsion tensor T^N. Let a second smooth manifold M and a smooth map $f : M \to N$ be given. The differential $F := df : TM \to f^*TN = E$ of the map f induces a vector bundle homomorphism between the tangent bundle TM of M and the pull-back of TN via f. The bundle homomorphism $T^E : \Lambda^2 E \to E$ induced by the torsion T^N of N satisfies the identity

$$\nabla^E_V F(W) - \nabla^E_W F(V) - F([V,W]) = T^E(F(V), F(W)), \tag{6.2}$$

where $\nabla^E = f^*\nabla^N$ denotes the pull-back connection, i.e. the connection which is induced on E by ∇^N and where $V, W \in \Gamma(TM)$.
In the rest of this section we denote by D a nice connection on the almost para-complex manifold (M,τ). Under this assumption we restate the condition (6.2):

$$\begin{array}{rcl} T^E(F(V),F(W)) & = & \nabla^E_V F(W) - \nabla^E_W F(V) - F([V,W]) \\ & = & \nabla^E_V F(W) - \nabla^E_W F(V) \\ & - & F(D_V W) + F(D_W V) + F(T(V,W)) \\ & = & \nabla^E_V F(W) - \nabla^E_W F(V) \\ & - & F(D_V W) + F(D_W V) + \lambda F(N_\tau(V,W)) \\ & = & (\nabla_V F)W - (\nabla_W F)V + \lambda F(N_\tau(V,W)), \end{array} \tag{6.3}$$

where ∇ is the connection induced on $T^*M \otimes E$ by D and ∇^E.
Later in this work we consider the case where N is a pseudo-Riemannian symmetric space with its Levi-Civita connection ∇^N.
Let $\alpha \in \mathbb{R}$ and define $\mathcal{R}_\alpha : TM \to TM$ as

$$\mathcal{R}_\alpha(X) = \cosh(\alpha)X + \sinh(\alpha)\tau X.$$

This is a parallel endomorphism field on the tangent bundle TM of M. The eigenvalues of which are seen to be $e^{e\alpha}$ on $T^{1,0}M$ and $e^{-e\alpha}$ on $T^{0,1}M$.
An associated family for f is a family of maps $f_\alpha : M \to N$ with $\alpha \in \mathbb{R}$ and $f_0 = f$, such that

$$\Phi_\alpha \circ df_\alpha = df \circ \mathcal{R}_\alpha, \quad \forall \alpha \in \mathbb{R}, \tag{6.4}$$

for some bundle isomorphism $\Phi_\alpha : f_\alpha^* TN \to f^*TN, \alpha \in \mathbb{R}$, which is parallel with respect to ∇^N, i.e. Φ_α satisfies

$$\Phi_\alpha \circ (f_\alpha^* \nabla^N) = (f^*\nabla^N) \circ \Phi_\alpha.$$

One can check, that each map f_α of an associated family itself admits an associated family.

Theorem 5 *Let (M,τ) be an almost para-complex manifold endowed with a nice connection D, N be a smooth manifold endowed with a torsion-free connection ∇^N and $f : (M,D,\tau) \to (N,\nabla^N)$ be a smooth map admitting an associated family f_α, then f is para-pluriharmonic. More precisely, each map of the associated family f_α is para-pluriharmonic.*

Proof: Since Φ_α is parallel with respect to ∇^N, ∇^N is torsion-free and D is nice, we are able to apply equation (6.3) to the family $df_\alpha = F_\alpha = \Phi_\alpha^{-1} \circ df \circ \mathcal{R}_\alpha$ and find

$$(\nabla_V F_\alpha)W - (\nabla_W F_\alpha)V + \lambda F_\alpha(N_\tau(V,W)) = 0.$$

As $\mathcal{R}_\alpha$ is D-parallel we obtain

$$(\nabla_X F_\alpha) = \Phi_\alpha^{-1} \circ (\nabla_X F) \circ \mathcal{R}_\alpha.$$

If $Z = X - e\tau X$ and $W = Y + e\tau Y$ have different type it holds $N_\tau(Z, W) = 0$, where we extended the Nijenhuis tensor para-complex linearly. This yields

$$(\nabla_V F_\alpha)W = (\nabla_W F_\alpha)V,\ \forall \alpha \in \mathbb{R}$$

and further we get

$$\begin{aligned}
(\nabla_Z F_\alpha)W &= e^{e\alpha}\,\Phi_\alpha^{-1}(\nabla_Z F)W, \\
(\nabla_W F_\alpha)Z &= e^{-e\alpha}\,\Phi_\alpha^{-1}(\nabla_W F)Z = e^{-e\alpha}\,\Phi_\alpha^{-1}(\nabla_Z F)W,
\end{aligned}$$

for all $\alpha \in \mathbb{R}$. Since this should coincide, it follows $(\nabla df)^{(1,1)} = 0$, i.e. $f : (M, D, \tau) \to (N, \nabla^N)$ is para-pluriharmonic. The rest follows, since each map of the associated family f_α admits an associated family $g_\beta = f_{(\alpha+\beta)}$. □

This is the motivation of the following definition:

Definition 11 *Let (M, τ) be an almost para-complex manifold endowed with a nice connection D, N be a smooth manifold endowed with a torsion-free connection ∇^N. A smooth map $f : (M, D, \tau) \to (N, \nabla^N)$ is said to be* $\tilde{\mathbb{S}}^1$-pluriharmonic *if and only if it admits an associated family.*

Given a para-hermitian metric g on M then in general a nice connection D is not the Levi-Civita connection ∇^g of g. Therefore the para-pluriharmonic equation (6.1) does not imply the harmonicity of f. If in addition the tensor $D - \nabla^g$ is trace-free the para-pluriharmonic equation implies the harmonic equation. This is true in the case of a special para-Kähler manifold (M, τ, g, ∇) and for a nearly para-Kähler manifold, where $D = \overline{\nabla}$ and $\overline{\nabla} - \nabla^g$ is skew-symmetric.

Proposition 7 *Let (M, τ, g) be an almost para-hermitian manifold endowed with a nice connection D, N be a pseudo-Riemannian manifold with its Levi-Civita connection ∇^N. Suppose, that the tensor $S = \nabla^g - D$ is trace-free. Then a para-pluriharmonic map $f : M \to N$ is harmonic.*

Proof: We calculate the expression

$$\begin{aligned}
\mathrm{tr}\,_g(\nabla df) &= \sum_i g(e_i, e_i)\left[\nabla^E_{e_i} df(e_i) - df(D_{e_i} e_i)\right] \\
&= \sum_i g(e_i, e_i)\left[\nabla^E_{e_i} df(e_i) - df((\nabla^g - S)_{e_i} e_i)\right] \\
&= \sum_i g(e_i, e_i)\left[\nabla^E_{e_i} df(e_i) - df(\nabla^g_{e_i} e_i)\right] \\
&= \mathrm{tr}\,_g(\tilde{\nabla}^g df)
\end{aligned}$$

where $\tilde{\nabla}^g$ is the connection induced on $T^*M \otimes E$ by ∇^g and ∇^E and e_i is an orthogonal basis for g on TM. Since g is para-hermitian, we obtain from the para-pluriharmonic equation

$$\mathrm{tr}\,_g(\nabla df) = \mathrm{tr}\,_g(\nabla df^{(1,1)}) = 0.$$

□

7 Related para-pluriharmonic and harmonic maps

7.1 The classifying map of a flat nearly para-Kähler manifold

In this section we consider simply connected almost para-hermitian manifolds (M, τ, g) endowed with a flat metric connection ∇ such that (∇, τ) satisfies the nearly para-Kähler condition.
In particular, simply connected flat nearly para-Kähler manifolds (M^{2n}, τ, g), i.e. nearly para-Kähler manifolds (M, τ, g) with flat Levi-Civita connection ∇^g are of this type.
Since (M, g, ∇) is simply connected and flat, we may identify by fixing a ∇-parallel frame s_0 its tangent bundle TM with $(M \times V, \langle \cdot, \cdot \rangle)$, where $V = C^n = (\mathbb{R}^{2n}, j_0)$ is endowed with the standard scalar product $\langle \cdot, \cdot \rangle$ of signature (n, n) on $V = \mathbb{R}^{2n}$.
The compatible para-complex structure τ defines via this identification a map

$$\tau : M \to \mathcal{P}(V, \langle \cdot, \cdot \rangle),$$

where $\mathcal{P}(V, \langle \cdot, \cdot \rangle)$ is the set of para-complex structures on V which are compatible with $\langle \cdot, \cdot \rangle$ and the orientation.

One can consider $\mathcal{P}(V, \langle \cdot, \cdot \rangle)$ as a subset in the vector space $\mathfrak{so}(n, n) = \mathfrak{so}(V) \subset \mathrm{Mat}(\mathbb{R}^{2n})$ characterised by the equation

$$f(j) = \mathbb{1}_{2n}, \tag{7.1}$$

where $f : \mathrm{Mat}(\mathbb{R}^{2n}) \to \mathrm{Mat}(\mathbb{R}^{2n})$ is given by $f : A \mapsto A^2$. The differential of this map is $df_A(H) = \{A, H\}$ for $A, H \in \mathrm{Mat}(\mathbb{R}^{2n})$. We remark, that elements satisfying the equation (7.1) define automatically para-complex structures, since they are trace-free and hence their eigenspaces to the eigenvalues ± 1 have the same dimension. The differential df has constant rank in points j satisfying the equation (7.1), since one sees

$$\begin{aligned} \ker df_j &= \{A \in \mathfrak{so}(V) \,|\, \{j, A\} = 0\}, \\ \mathrm{im}\, df_j &\cong \{A \in \mathfrak{so}(V) \,|\, [j, A] = 0\} \cong \mathfrak{u}^\pi(C^n). \end{aligned}$$

Applying the regular value theorem $\mathcal{P}(V, \langle \cdot, \cdot \rangle)$ is shown to be a submanifold of $\mathfrak{so}(V)$. Its tangent space at $j \in \mathcal{P}(V, \langle \cdot, \cdot \rangle)$ is

$$T_j\, \mathcal{P}(V, \langle \cdot, \cdot \rangle) = \ker df_j = \{A \in \mathfrak{so}(V) \,|\, \{j, A\} = 0\}. \tag{7.2}$$

Moreover, $\mathcal{P}(V, \langle \cdot, \cdot \rangle)$ can be identified with the pseudo-Riemannian symmetric space $SO_0(n, n)/U^\pi(C^n)$, where $SO_0(n, n)$ is the identity component of the special pseudo-orthogonal group $SO(n, n)$ and $U^\pi(C^n)$ is the para-unitary group, by the map

$$\begin{aligned} \Phi \;\; : \;\; & SO_0(n, n)/U^\pi(C^n) \to \mathcal{P}(V, \langle \cdot, \cdot \rangle), \\ & gK \mapsto g\, j_0\, g^{-1}, \end{aligned}$$

which maps the canonical base point $o = eK$ to j_0.
An element $j \in \mathcal{P}(V, \langle \cdot, \cdot \rangle)$ defines a symmetric decomposition of $\mathfrak{so}(V)$ by

$$\begin{aligned} \mathfrak{p}(j) &= \{A \in \mathfrak{so}(V) \,|\, \{j, A\} = 0\}, \\ \mathfrak{k}(j) &= \{A \in \mathfrak{so}(V) \,|\, [j, A] = 0\} \cong \mathfrak{u}^\pi(C^n). \end{aligned}$$

In particular we note $\mathfrak{k}(j_0) = \mathfrak{u}^\pi(C^n)$. Moreover, one observes $T_j\, \mathcal{P}(V, \langle\cdot,\cdot\rangle) = \mathfrak{p}(j)$.
Let $\tilde{j} \in SO_0(n,n)/U^\pi(C^n)$ and $j = \Phi(\tilde{j})$, then $T_{\tilde{j}} SO_0(n,n)/U^\pi(C^n)$ is canonically identified with $\mathfrak{p}(j)$. We determine now the differential of the above identification:

Proposition 8 *Let $\Psi = \Phi^{-1} : \mathcal{P}(V, \langle\cdot,\cdot\rangle) \to SO_0(n,n)/U^\pi(C^n)$. Then it holds at $j \in \mathcal{P}(V, \langle\cdot,\cdot\rangle)$*

$$d\Psi : T_j\, \mathcal{P}(V, \langle\cdot,\cdot\rangle) \ni X \mapsto -\frac{1}{2} j^{-1} X \in \mathfrak{p}(j). \tag{7.3}$$

The next theorem constructs the desired para-pluriharmonic map.

Theorem 6 *Let (M, τ, g) be a simply connected almost para-hermitian manifold endowed with a flat metric connection ∇ such that (∇, τ) satisfies the nearly para-Kähler condition, then $(TM, D = \nabla - S, S = -\frac{1}{2}\tau(\nabla\tau), \omega = g(\tau\cdot,\cdot))$ defines a symplectic ptt*-bundle and the matrix of τ in a D^θ-flat frame $s^\theta = (s_i^\theta)$ defines an $\tilde{\mathbb{S}}^1$-pluriharmonic map $\tilde{\tau}^\theta : M \to \mathcal{P}(V, \langle\cdot,\cdot\rangle) \to SO_0(n,n)/U^\pi(C^n)$.*
In particular, given a nice connection D on M the map

$$\tilde{\tau}^\theta : (M, \tau, D) \to SO_0(n,n)/U^\pi(C^n)$$

is para-pluriharmonic.

Proof: One observes $D^\theta g = 0$, as one has $\nabla g = 0$ and $S_X^\theta := \cosh(\theta) S_X + \sinh(\theta) S_{\tau X}$ takes values in $\mathfrak{so}(V)$. This means we can choose for each θ the D^θ-flat frame s^θ orthonormal, such that $s^{\theta=0} = s_0$. Using $D\tau = 0$ (compare theorem 1 and lemma 1) we obtain

$$Xg(\tau s_i^\theta, s_j^\theta) = g(D_X^\theta(\tau s_i^\theta), s_j^\theta) = g((D_X^\theta \tau) s_i^\theta, s_j^\theta) = g([S_X^\theta, \tau] s_i^\theta, s_j^\theta) = -2g(\tau S_X^\theta s_i^\theta, s_j^\theta).$$

Let S^{s^θ} and τ^{s^θ} be the representation of S and τ in the frame s^θ, then we have

$$(\tau^{s^\theta})^{-1} X(\tau^{s^\theta}) = -2 S^{s^\theta}$$

or

$$d\tilde{\tau}^\theta = (s^\theta)^{-1} \circ S^\theta \circ s^\theta.$$

Here the frame s^θ is seen as a map $s^\theta : M \times V \to TM$. For $X \in \Gamma(TM)$ we obtain

$$\begin{aligned} d\tilde{\tau}^\theta(X) &= (s^\theta)^{-1} \circ S_X^\theta \circ (s^\theta) = (s^\theta)^{-1} \circ S_{\mathcal{R}_\theta X} \circ (s^\theta) \\ &= ((s^\theta)^{-1} s^0) \circ d\tilde{\tau}(\mathcal{R}_\theta X) \circ ((s^0)^{-1} s^\theta) \\ &= Ad_{\alpha_\theta}^{-1} \circ d\tilde{\tau}(\mathcal{R}_\theta X) = \Phi_\theta^{-1} \circ d\tilde{\tau}(\mathcal{R}_\theta X), \end{aligned}$$

where $\alpha_\theta = (s^\theta)^{-1} s^0$ is the frame change from s_0 to s_θ and $\Phi_\theta = Ad_{\alpha_\theta}$. Φ_θ is parallel with respect to the Levi-Civita connection on $SO_0(n,n)/U^\pi(C^n)$ and hence $\tilde{\tau}^\theta$ is $\tilde{\mathbb{S}}^1$-pluriharmonic. Given a nice connection D on M theorem 5 shows that $\tilde{\tau}^\theta$ is para-pluriharmonic. □

The next corollary emphasises the nearly para-Kähler setting:

Corollary 3 *Let (M, τ, g) be a simply connected flat nearly para-Kähler manifold and $(TM, \overline{\nabla} = \nabla^g - S, S = -\frac{1}{2}\tau(\nabla\tau), \omega(\cdot,\cdot) = g(\tau\cdot,\cdot))$ the associated symplectic ptt*-bundle, then the matrix of τ in a D^θ-flat frame $s^\theta = (s^\theta_i)$ defines an $\tilde{\mathbb{S}}^1$-pluriharmonic map $\tilde{\tau}^\theta : M \to \mathcal{P}(V, \langle\cdot,\cdot\rangle) \to SO_0(n,n)/U^\pi(C^n)$.*

For nearly para-Kähler manifolds one can get more precise information about the map $\tilde{\tau}^\theta$:

Theorem 7 *Let (M, τ, g) be a simply connected flat nearly para-Kähler manifold and $(TM, \overline{\nabla} = \nabla^g - S, S = -\frac{1}{2}\tau(\nabla\tau), \omega(\cdot,\cdot) = g(\tau\cdot,\cdot))$ the associated symplectic ptt*-bundle. Then the connection $\overline{\nabla}$ is nice and the matrix of τ in a D^θ-flat frame $s^\theta = (s^\theta_i)$ defines a para-pluriharmonic map $\tilde{\tau}^\theta : (M, \tau, \bar{\nabla}) \to \mathcal{P}(V, \langle\cdot,\cdot\rangle) \to SO_0(n,n)/U^\pi(C^n)$. Moreover, the map $\tilde{\tau}^\theta$ is harmonic.*

Proof: We have to prove, that $\overline{\nabla}$ is nice. Therefore we rewrite the Nijenhuis tensor

$$\begin{aligned} N_\tau(X,Y) &= (\nabla_{\tau X}\tau)Y - (\nabla_{\tau Y}\tau)X - \tau(\nabla_X\tau)Y + \tau(\nabla_Y\tau)X \\ &= -4\tau(\nabla_X\tau)Y, \end{aligned}$$

where the second equality follows from the nearly para-Kähler condition and by

$$(\nabla_{\tau X}\tau)Y = -(\nabla_Y\tau)\tau X = \tau(\nabla_Y\tau)X = -\tau(\nabla_X\tau)Y.$$

On the other hand the torsion of $\overline{\nabla}$ was given in equation (5.7) by

$$T^{\overline{\nabla}}(X,Y) = \tau(\nabla_X\tau)Y$$

and consequently $\overline{\nabla}$ is nice.
Due to corollary 3 the map $\tilde{\tau}^\theta$ is $\tilde{\mathbb{S}}^1$-pluriharmonic. Since $\overline{\nabla}$ is nice, theorem 5 implies that $\tilde{\tau}^\theta$ is para-pluriharmonic. From the skew-symmetry of S and proposition 7 we obtain that $\tilde{\tau}^\theta$ is harmonic. □

7.2 The dual Gauß map of a special para-Kähler manifold with torsion

In this subsection we consider a simply connected almost para-hermitian manifold (M, τ, g) with a flat connection ∇, such that (∇, τ) is special and the two-form $\omega = g(\tau\cdot,\cdot)$ is ∇-parallel.
Using the flat connection ∇ we identify by fixing a ∇-parallel symplectic frame s_0 the tangent bundle (TM, ω) with $(M \times V, \omega_0)$ where $V = \mathbb{R}^{2n}$ and ω_0 is its standard symplectic form.
The compatible para-complex structure τ is seen as a map

$$\tau : M \to \mathcal{P}(V, \omega_0),$$

where $\mathcal{P}(V, \omega_0)$ is the set of para-complex structures on V which are compatible with ω_0.
Now we discuss the differential geometry of $\mathcal{P}(V, \omega_0)$, where ω_0 is the standard symplectic

form on $V = C^n = (\mathbb{R}^{2n}, j_0)$.
First, we consider $\mathcal{P}(V, \omega_0)$ as a subset of the vector space $\mathfrak{sp}(\mathbb{R}^{2n}) \subset \mathrm{Mat}(\mathbb{R}^{2n})$ characterised by the set of equations

$$f(j) = \mathbb{1}_{2n}, \tag{7.4}$$

where $f : \mathrm{Mat}(\mathbb{R}^{2n}) \to \mathrm{Mat}(\mathbb{R}^{2n})$ is given as in the last section. Again, df has constant rank in points j satisfying the equation (7.4), since one sees

$$\begin{aligned} \ker df_j &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, \{j, A\} = 0\}, \\ \operatorname{im} df_j &\cong \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, [j, A] = 0\} \cong \mathfrak{u}^\pi(C^r). \end{aligned}$$

Applying the regular value theorem we obtain that $\mathcal{P}(V, \omega_0)$ is a submanifold of $\mathfrak{sp}(\mathbb{R}^{2n})$. Its tangent space at $j \in \mathcal{P}(V, \omega_0)$ is

$$T_j\, \mathcal{P}(V, \omega_0) = \ker df_j = \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, \{j, A\} = 0\}. \tag{7.5}$$

In addition the manifold $\mathcal{P}(V, \omega_0)$ can be identified with the pseudo-Riemannian symmetric space $\mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$ by the map

$$\begin{aligned} \Phi \,:\, \mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n) &\to \mathcal{P}(V, \omega_0), \\ gK &\mapsto g\, j_0\, g^{-1}, \end{aligned}$$

which maps the canonical base point $o = eK$ to j_0.
Any $j \in \mathcal{P}(V, \omega_0)$ defines a symmetric decomposition of $\mathfrak{sp}(\mathbb{R}^{2n})$ by

$$\begin{aligned} \mathfrak{p}(j) &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, \{j, A\} = 0\}, \\ \mathfrak{k}(j) &= \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \,|\, [j, A] = 0\} \cong \mathfrak{u}^\pi(C^r). \end{aligned}$$

In particular it is $\mathfrak{k}(j_0) = \mathfrak{u}^\pi(C^r)$. Moreover, one observes $T_j\, \mathcal{P}(V, \omega_0) = \mathfrak{p}(j)$.
Let $\tilde{j} \in \mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$ and $j = \Phi(\tilde{j})$, then $T_{\tilde{j}}\mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$ is canonically identified with $\mathfrak{p}(j)$ and for the differential of the identification we have:

Proposition 9 *Let $\Psi = \Phi^{-1} : \mathcal{P}(V, \omega_0) \to \mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$. Then it holds at $j \in \mathcal{P}(V, \omega_0)$*

$$d\Psi : T_j\, \mathcal{P}(V, \omega_0) \ni X \mapsto -\frac{1}{2} j^{-1} X \in \mathfrak{p}(j). \tag{7.6}$$

Recall, that under the above assumptions $(TM, D = \nabla - S, S = -\frac{1}{2}\tau(\nabla\tau), g)$ defines a metric *ptt**-bundle. Analogue to the last section we obtain:

Theorem 8 *Let (M, τ, g) be a simply connected almost para-hermitian manifold endowed with a flat connection ∇, such that (∇, τ) is special and the two-form $\omega = g(\tau\cdot, \cdot)$ is ∇-parallel and let $(TM, D = \nabla - S, S = -\frac{1}{2}\tau(\nabla\tau), g)$ be the associated metric ptt*-bundle. Then the matrix of τ in a D^θ-flat frame $s^\theta = (s_i^\theta)$ defines an $\tilde{\mathbb{S}}^1$-pluriharmonic map $\tilde{\tau}^\theta : M \to \mathcal{P}(V, \omega_0) \to \mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$.*
In particular, given a nice connection D on (M, τ) then the map $\tilde{\tau}^\theta : (M, \tau, D) \to \mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$ is para-pluriharmonic.

Proof: Since we have $D^0\omega = \nabla\omega = (D+S)\omega = 0$ and $S^\theta_X := \cosh(\theta)S_X + \sinh(\theta)S_{\tau X}$ is skew-symmetric with respect to ω, we obtain $D\omega = 0$ and $D^\theta\omega = 0$. Hence one can choose for each θ the D^θ-parallel frame s^θ as a symplectic frame, such that $s^{\theta=0} = s_0$. This yields using $D\tau = 0$ (cf. theorem 1 and lemma 1)

$$X.\omega(\tau s^\theta_i, s^\theta_j) = \omega(D^\theta_X(\tau s^\theta_i), s^\theta_j) = \omega((D^\theta_X\tau)s^\theta_i, s^\theta_j) = \omega([S^\theta_X,\tau]s^\theta_i, s^\theta_j) = -2\omega(\tau S^\theta_X s^\theta_i, s^\theta_j).$$

Denote by $S^{s^\theta}, \tau^{s^\theta}$ the representation of S and τ in the frame s^θ, then we get

$$(\tau^{s^\theta})^{-1}X(\tau^{s^\theta}) = -2S^{s^\theta}$$

or

$$d\tilde{\tau}^\theta = (s^\theta)^{-1} \circ S^\theta \circ s^\theta,$$

where the frame s^θ is seen as a map $s^\theta : M \times V \to TM$. This shows for $X \in \Gamma(TM)$

$$\begin{aligned} d\tilde{\tau}^\theta(X) &= (s^\theta)^{-1} \circ S^\theta_X \circ (s^\theta) = (s^\theta)^{-1} \circ S_{\mathcal{R}_\theta X} \circ (s^\theta) \\ &= ((s^\theta)^{-1}s^0) \circ d\tilde{\tau}(\mathcal{R}_\theta X) \circ ((s^0)^{-1}s^\theta) \\ &= Ad^{-1}_{\alpha_\theta} \circ d\tilde{\tau}(\mathcal{R}_\theta X) = \Phi^{-1}_\theta \circ d\tilde{\tau}(\mathcal{R}_\theta X), \end{aligned}$$

where $\alpha_\theta = (s^\theta)^{-1}s^0$ denotes the frame change from s_0 to s_θ and $\Phi_\theta = Ad_{\alpha_\theta}$ which is parallel with respect to the Levi-Civita connection on $\mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$. In other words we have found an associated family. Given a nice connection D on (M,τ) theorem 5 shows that $\tilde{\tau}^\theta$ is para-pluriharmonic. $\square$

In the case where the above *ptt**-bundle comes from a special para-Kähler manifold we have the theorem:

Theorem 9 *Let (M,τ,g,∇) be a simply connected special para-Kähler manifold and $(TM, D = \nabla - S, S = -\frac{1}{2}\tau\nabla\tau, g)$ be the associated metric ptt*-bundle, then the matrix of τ in a D^θ-flat frame $s^\theta = (s^\theta_i)$ defines a para-pluriharmonic map $\tilde{\tau}^\theta : (M,\tau,D) \to$ $\mathrm{Sp}(\mathbb{R}^{2n})/U^\pi(C^n)$. Further $\tilde{\tau}^\theta$ is harmonic.*

Proof: Using theorem 8 the map $\tilde{\tau}^\theta$ is $\tilde{\mathbb{S}}^1$-pluriharmonic. In the special para-Kähler case we know that D is the Levi-Civita connection and hence it is torsion-free. The para-complex structure τ is integrable and hence it holds $N_\tau = 0$. This means, that D is nice and theorem 5 shows that $\tilde{\tau}^\theta$ is para-pluriharmonic. Since S is trace-free we get from proposition 7 that $\tilde{\tau}^\theta$ is harmonic. $\square$

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